UDC 62-50

POSITION CONTROL IN PARABOLIC SYSTEMS

PMM Vol. 41, № 2, 1977, pp. 195-201

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(Received November 25, 1976)

Problems of controlling parabolic systems under conditions of uncertainty are studied. Necessary and sufficient conditions are indicated for the solvability of the problems and methods for constructing the required controls are given. The paper is closely related to the researches in [1 — 9]. The main difference in the present paper, as also in [4 — 8], from the known researches on control problems for distributed-parameter systems (see [10 — 15], for instance) is that here the principle of feed-back control is discussed.

1. Consider a system whose state at each instant t from a specified time interval $[t_0, \vartheta]$ is characterized by the scalar function $y(t, \cdot) = y(t, x)$ defined in a domain Ω of an n-dimensional Euclidean space R^n. The system is subject to controls u_1 and u_2 and to uncontrolled disturbances v_1 and v_2. The system's dynamics is described by the equation and collection of boundary conditions

$$\partial y(t, x)/\partial t = Ay(t, x) + b_1(t, x)u_1(t, x) - c_1(t, x)v_1(t, x) + f(t, x);\; x \in \Omega,\; t_0 < t \leqslant \vartheta \tag{1.1}$$

$$\sigma_1 \frac{\partial y(t, x)}{\partial \nu_A} + \sigma_2(x)\, y(t, x) = b_2(x)\, u_2(t) - c_2(x)v_2(t);\; x \in \Gamma,\; t_0 < t \leqslant \vartheta \tag{1.2}$$

$$y(t_0, x) = y_0(x), \quad x \in \Omega \tag{1.3}$$

$$Ay = \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right) + a(x)\, y$$

Here Γ is the boundary of Ω; $\partial / \partial \nu_A$ is the conormal derivative. At each instant t the controls are subject to the constraints $u_1(t, \cdot) \in P_1(t)$ and $u_2(t) \in P_2(t)$, where $P_1(t)$ is some collection of functions defined on Ω with values in R^{r_1}; $P_2(t) \subset R^{r_2}$; at each instant t we have the estimates $v_1(t, \cdot) \in Q_1(t)$ and $v_2(t) \in Q_2(t)$ for the disturbances v_1 and v_2, where $Q_1(t)$ is some collection of functions defined on Ω with values in R^{m_1}; $Q_2(t) \subset R^{m_2}$.

The main purpose of the paper is to study the following problem to which reduce many standard problems of the conflict control of system (1.1), (1.2). Under the specified constraints on the resources of control u_1 and u_2 and for known estimates on the intensities of disturbances v_1 and v_2 we are required to find a method for forming the controls u_1 and u_2 on the feed-back principle $(u_1[t, x] = u_1(t, x, y[t, \cdot])$ and $u_2[t] = u_2(t, y[t, \cdot]))$, which, for any admissible realizations of the disturbances, would guarantee that system (1.1) — (1.3) is led onto a specified state set in a specified period of time, and in such a way that specified phase constraints are satisfied during the control.

Individual versions of the problem for system (1.1), (1.2) were studied in [5 — 7]. Thus, the case when the boundary condition (1.2) is a homogeneous Dirichlet condition ($\sigma_1 = 0$, $b_2 = 0$ and $c_2 = 0$) and there are no phase constraints on the system's states, in particular, was considered (*) from the viewpoint of the theory of semigroups in [5]. A version of the problem was studied in [6] when only the boundary of domain Ω is subject to the controls and to the disturbances ($b_1 = 0$ and $c_1 = 0$), where relation (1.2) once again is the Dirichlet boundary condition and we are dealing with the task of leading the system onto a specified set of states at a specified instant in the absence of phase constraints. The effectiveness of the construction of the control procedure proposed in [5] was discussed in [7]. Below we consider the problem for system (1.1), (1.2) in a general formulation.

2. Let us refine the statement of the problem. The symbol $B(E_1; E_2)$ denotes the Banach space B of functions on E_1 with values in E_2; $B(E_1) = B(E_1; R^1)$; $\|\cdot\|_B$ is the norm in B. Measurability and integrability are always understood in the Lebesgue measure sense and derivatives in the generalized sense (see [16 – 18], for example). We assume the satisfaction of the following conditions: Ω is a bounded domain with properties (1), (2) and R (see pp. 212 and 222 in [16]); A is a selfadjoint elliptic operator [16]; the functions a_{ij} and a are measurable and bounded on Ω, and $\partial a_{ij} / \partial x_k$ are bounded on Ω; b_1 and c_1 are measurable and bounded on $Q = (t_0, \vartheta) \times \Omega$; $f \in L^2(Q)$; b_2, c_2, and σ_2 are measurable and bounded on Γ and $\sigma_1 \cdot \sigma_2 \geqslant 0$ in Γ; $\sigma_2 > 0$ when $\sigma_1 = 0$; $y_0 \in L^2(\Omega)$. Further, the sets

$$P_1(t) \subset L^2(\Omega; R^{r_1}),\ P_2(t) \subset R^{r_2},\ Q_1(t) \subset L^2(\Omega; R^{m_1}),\ Q_2(t) \subset R^{m_2},$$

and in the appropriate spaces these sets are convex, closed, measurable and equibounded with respect to $t \in [t_0, \vartheta]$. All the quantities to be examined are real.

Let $u_i(\cdot) = u_i(t)$, $t_1 \leqslant t < t_2$ be a measurable function with values in $P_i(t)$, $i = 1, 2$. According to the theorems on a measurable selection [19] such functions exist. Every pair $u(\cdot) = u(\cdot; t_1, t_2) = \{u_1(\cdot), u_2(\cdot)\}$ of such functions is called a u-program. The rule $U = U(t_1, t_2, y)$ which associates some program $u(\cdot; t_1, t_2)$ with each triple $\{t_1, t_2, y\}$, where $t_1 \in [t_0, \vartheta)$, $t_2 \in (t_1, \vartheta]$ and $y \in L^2(\Omega)$, (so that $U(t_1, t_2, y) = u(\cdot; t_1, t_2)$) is called a strategy U.

Let us introduce the concept of the motion of system (1.1), (1.2), corresponding to strategy U. Let Δ be a finite partitioning of $[t_0, \vartheta]$ by the points $t_0 = \tau_0 < \tau_1 < \ldots < \tau_{m(\Delta)} = \vartheta$, $\delta(\Delta) = \max_i (\tau_{i+1} - \tau_i)$. We form the sets

$$\Phi = \{\varphi \in H^{2,1}(Q) | \varphi(\vartheta, x) = 0,\ x \in \Omega;$$

$$\sigma_1 \partial\varphi(t, x) / \partial\nu_A + \sigma_2(x)\varphi(t, x) = 0,\ x \in \Gamma,\ t \in (t_0, \vartheta)\}$$

Here $H^{2,1}(Q)$ is a Sobolev space consisting of all elements of $L^2(Q)$ having first- and second-order generalized derivatives in x and first-order in t. Every function $y[\cdot]_\Delta \in C([t_0, \vartheta]; L^2(\Omega))$ satisfying the equality

$$\int_Q y\left(-\frac{\partial\varphi}{\partial t} - A\varphi\right) dx\, dt = \int_Q (f + b_1 u_1[t] - \qquad (2.1)$$

*) Similar questions were discussed also by: Osipov, Iu. S., Differential games in distributed-parameter systems. Abstracts Third All-Union Conf. on the Theory of Games, Odessa, 1974.

$$c_1 v_1 [t]) \varphi\, dx\, dt + \int_\Omega y_0 \varphi(t_0, x)\, dx +$$

$$\int_{t_0}^{\vartheta} \int_\Gamma (b_2 u_2 [t] - c_2 v_2 [t]) F(\varphi)\, d\Gamma\, dt$$

for any $\varphi \in \Phi$ is called a motion $y[t] = y[t]_\Delta = y[t;\ t_0,\ y_0,\ U]_\Delta,\ t_0 \leqslant t \leqslant \vartheta$ of system (1.1), (1.2) from a position $\{t_0,\ y_0\}$, corresponding to strategy U. Here $F(\varphi) = \varphi / \sigma_1$ when $\sigma_1 \neq 0$ and $F(\varphi) = -\sigma_2^{-1} \partial\varphi / \partial\nu_A$ when $\sigma_1 = 0$; on each interval $[\tau_i,\ \tau_{i+1})$

$$\{u_1[\cdot],\ u_2[\cdot]\} = U(\tau_i,\ \tau_{i+1},\ y[\tau_i])$$

and $v_1[t]$ and $v_2[t]$, $t_0 \leqslant t \leqslant \vartheta$ are some measurable realizations of the disturbances with values in $Q_1(t)$ and $Q_2(t)$, respectively. The existence of motions can be verified, for example, by Galerkin's method or by using the transposition method [9, 18] and relying on the theory of homogeneous boundary-value problems for equations of parabolic type [17].

The initial control problem can now be formulated as follows. Let M and N be certain sets in the space $[t_0,\ \vartheta] \times L^2(\Omega)$.

Problem 2.1. Construct a strategy U with the property: for any number $\varepsilon > 0$ a number $\delta > 0$ can be found such that the condition

$$\rho(\{t_y,\ y[t_y]_\Delta\},\ M) = \inf_{\{t_*,\ h_*\} \in M} (|t_y - t_*|^2 + \|y[t_y]_\Delta - h_*\|_{L^2(\Omega)})^{1/2} \leqslant \varepsilon \qquad (2.2)$$

where

$$\rho(\{t,\ y[t]_\Delta\},\ N) \leqslant \varepsilon, \quad t_0 \leqslant t \leqslant t_y \qquad (2.3)$$

is satisfied at the instant $t_y = t(y[\cdot]_\Delta)$ for every motion $y[t]_\Delta = y[t;\ t_0,\ y_0,\ U]_\Delta$ with $\delta(\Delta) \leqslant \delta$.

We indicate the conditions for the solvability of Problem 2.1 and a method for constructing the strategy required.

3. Let $\{\lambda_j,\ \omega_j\}$ be a solution in $H^1(\Omega)$ of the spectral problem

$$A\omega = -\lambda\omega, \quad x \in \Omega; \quad \sigma_1 \frac{\partial\omega}{\partial\nu_A} + \sigma_2 \omega|_\Gamma = 0$$

where $H^m(\Omega)$ is a Sobolev space of order m in domain Ω (see [10, 17, 18]). Let $\{\alpha_j;\ j = 1, 2, \ldots\}$ be some set of numbers α_j satisfying the conditions: if $\sigma_1 \neq 0$, then $\alpha_j = 1,\ j = 1, 2, \ldots$; if $\sigma_1 = 0$ then

$$0 < \alpha_j < 1, \quad \sum_{j=1}^{\infty} \alpha_j^2 < \infty, \quad \sum_{j=1}^{\infty} \alpha_j^2 \left\| \frac{\partial\omega_j}{\partial\nu_A} \right\|^2_{L^2(\Gamma)} < \infty$$

(Such numbers α_j exist since $\partial\omega_j / \partial\nu_A \in L^2(\Gamma)$ under the assumptions made on the system's parameters and on domain Ω, according to the imbedding theorem (see [16], for example)).

For $y \in L^2(\Omega)$ we denote

$$\|y\|_\alpha = \left(\sum_{j=1}^{\infty} \alpha_j^2 \langle y,\ \omega_j\rangle_{L^2(\Omega)}\right)^{1/2}$$

Here $\langle\cdot,\ \cdot\rangle_{L^2(\Omega)}$ is the symbol for the scalar product in $L^2(\Omega)$. When $\sigma_1 \neq 0$ we have $\|y\|_\alpha = \|y\|_{L^2(\Omega)}$. Let K be some set in the position space $[t_0,\ \vartheta] \times L^2(\Omega)$. By the symbol U^ε we denote a strategy of the following form. Suppose that some triple

$\{t_1, t_2, y\}$ has been selected. If $K(t_1) = \varnothing$, then $U^e(t_1, t_2, y)$ is an arbitrary program $u(\cdot; t_1, t_2)$. If $K(t_1) \neq \varnothing$, then $U^e(t_1, t_2, y) = \{u_1^e(\cdot), u_2^e(\cdot)\}$. Here the $u_\nu^e(\cdot)$, $\nu = 1, 2$ are functions with the property: sequences $\{u_\nu^{(k)}\}$, $\nu = 1, 2$ and $\{z^{(k)}\}$ exist such that

$$\lim_{k\to\infty} \| y - z^{(k)} \|_\alpha = \inf_{h\in K(t_1)} \| y - h \|_\alpha$$

$$z^{(k)} \in K(t_1), \quad k = 1, 2, \ldots$$

$$\{u_\nu^{(k)}\} \to u_\nu^e \text{ weakly in } L^2([t_1, t_2); R^{r_\nu})$$

$$\langle u_1^{(k)}, l_1(z^{(k)} - y)\rangle_{L^2([t_1, t_2); L^2(\Omega; R^{r_1}))} = \max_{\{u_1\}} \langle u_1, l_1(z^{(k)} - y)\rangle_{L^2([t_1, t_2); L^2(\Omega; R^{r_1}))}$$

$$\langle u_2^{(k)}, l_2(z^{(k)} - y)\rangle_{L^2([t_1, t_2); R^{r_2})} = \max_{\{u_2\}} \langle u_2, l_2(z^{(k)} - y)\rangle_{L^2([t_1, t_2); R^{r_2})}$$

$$l_1(y) = \sum_{j=1}^{\infty} \alpha_j^2 \langle y, \omega_j\rangle_{L^2(\Omega)} e^{-\lambda_j(2t_2-t_1-t)} \omega_j b_1(t, x)$$

$$l_2(y) = \sum_{j=1}^{\infty} \alpha_j^2 \langle y, \omega_j\rangle_{L^2(\Omega)} e^{-\lambda_j(2t_2-t_1-t)} \langle b_2, F(\omega_j)\rangle_{L^2(\Gamma)}$$

Here $K(t_1)$ is the section of K by the plane $t = t_1$; $\{u_\nu\}$, $\nu = 1, 2$ is the collection of all functions $u_\nu(t)$ measurable on $[t_1, t_2)$, with values in $P_\nu(t)$. The program $\{u_1^e(\cdot), u_2^e(\cdot)\}$ always exists under the condition $K(t_1) \neq \varnothing$. Strategy U^e is said to be extremal to set K.

The rule which associates a certain pair $v(\cdot; t_1, t_2) = \{v_1(\cdot), v_2(\cdot)\}$ of functions $v_1(t)$ and $v_2(t)$, measurable on the interval $[t_1, t_2)$ and having values in $Q_1(t)$ and $O_2(t)$, respectively, with every triple $\{t_1, t_2, y\}$ is called a strategy $V = V(t_1, t_2, y)$ (so that $V(t_1, t_2, y) = v(\cdot; t_1, t_2)$). Let $W(M, N)$ be the collection of all pairs $\{t_*, y_*\} \in [t_0, \vartheta] \times L^2(\Omega)$ with the property: for any strategy V and any numbers $\varepsilon > 0$ and $\delta > 0$ there exists at least one motion $y[t] = y[t]_\Delta = y[t; t_*, y_*, V]_\Delta$, $t_* \leqslant t \leqslant \vartheta$ with $\delta(\Delta) \leqslant \delta$, for which the condition

$$\rho(\{t_y, y[t_y]_\Delta\}, M) \leqslant \varepsilon$$

and

$$\rho(\{t, y[t]_\Delta\}, N) \leqslant \varepsilon, \quad t_* \leqslant t \leqslant t_y$$

is satisfied at some instant $t_y = t(y[\cdot]_\Delta) \in [t_*, \vartheta]$; (the motions $y[t; t_*, y_*, V]_\Delta$ are defined by analogy with the motions $y[t; t_*, y_*, U]_\Delta$: in the definition of the latter we need merely to replace the quantities U and $v_i[t]$ by the quantities V and $u_i[t]$, where now $u_i[t]$ are some measurable functions with values in $P_i(t)$, $i = 1, 2$).

Theorem 3.1. Let the sets M and N be closed in the metric

$$\|\{t, y\}\|_\alpha = (t^2 + \|y\|_\alpha^2)^{1/2} \tag{3.1}$$

Problem 2.1 is solvable if and only if

$$\{t_0, y_0\} \in W(M, N) \tag{3.2}$$

Under condition (3.2) the strategy U^e extremal to set $W(M, N)$ solves Problem 2.1.

The necessity of condition (3.2) for the solvability of Problem 2.1 is obvious. The assertion that under condition (3.2) the strategy U^e extremal to set W solves problem 2.1 can be verified in the following way. First of all we show (see the similar arguments in [1, 4]) that for each motion $y[t]_\Delta = y[t; t_0, y_0, U^e]_\Delta$ corresponding to a partitioning Δ with a sufficiently small diameter $\delta(\Delta)$, the point $\{t, y[t]_\Delta\}$ remains in a small neighborhood of W, measured in metric (3.1), until the instant it hits a small neighborhood of set M, also measured in metric (3.1). Therefore, for a sufficiently small $\delta(\Delta)$ each point $\{t, y[t]_\Delta\}$ necessarily rests, when $t \leqslant \vartheta$, in the preassigned neighborhood of M before leaving the preassigned neighborhood of set N; these neighborhoods are measured in metric (3.1). Taking into account that every bounded set in $L^2(\Omega)$ is compact in metric (3.1) and that the set of motions $y[\cdot; t_0, y_0, U^e]_\Delta$ is compact in the space $C([t_0, \vartheta]; L^2(\Omega))$; we also allow for the closedness of M and N in metric (3.1). Hence, relations (2.2) and (2.3) are satisfied for each motion $y[\cdot; t_0, y_0, U^e]_\Delta$ if only the diameter $\delta(\Delta)$ of partitioning Δ is fairly small.

Note 3.1. Let us consider the following evasion problem. Construct a strategy V with the property: numbers $\varepsilon > 0$ and $\delta > 0$ exist such that for every motion

$$y[t]_\Delta = y[t; t_0, y_0, V]_\Delta, \quad t_0 \leqslant t \leqslant \vartheta$$

with $\delta(\Delta) \leqslant \delta$, the condition that the instant $t_y = t(y[\cdot]_\Delta)$ for which

$$\rho(\{t, y[t]_\Delta\}, N) \leqslant \varepsilon, \quad t_0 \leqslant t \leqslant t_y$$

$$\rho(\{t_y, y[t_y]_\Delta\}, M) \leqslant \varepsilon$$

does not exist, is satisfied. Let sets M and N be closed in metric (3.1). It can be verified that the evasion problem has a solution if and only if the initial position $\{t_0, y_0\}$ does not satisfy condition (3.2). The strategy V solving this problem also can be constructed as a strategy extremal to some set in the space $[t_0, \vartheta] \times L^2(\Omega)$.

Note 3.2. Sets M and N are known to be closed in metric (3.1) if their projections onto space $L^2(\Omega)$ are bounded weakly-closed sets in $L^2(\Omega)$.

4. Let us show a case when the set $W(M, N)$ admits of an analytic description. We assume that $N = L^2(\Omega)$, set M lies wholly in the hyperplane $t = \vartheta$: $M = \{\{t, y\} | t = \vartheta, y \in M(\vartheta)\}$, and its section $M(\vartheta)$ is a convex bounded closed set in space $L^2(\Omega)$. We denote

$$G_0(t, \vartheta) y = \sum_{j=1}^{\infty} \omega_j(\cdot) e^{-\lambda_j(\vartheta - t)} \langle \omega_j, y \rangle_{L^2(\Omega)} \qquad (4.1)$$

$$G_1(t, \vartheta) f = \sum_{j=1}^{\infty} \omega_j(\cdot) \int_t^{\vartheta} e^{-\lambda_j(\vartheta - \tau)} \langle \omega_j, f \rangle_{L^2(\Omega)} d\tau$$

$$G_{1u}(t, \vartheta) u_1 = G_1(t, \vartheta) b_1 u_1, \quad G_{1v}(t, \vartheta) v_1 = G_1(t, \vartheta) c_1 v_1$$

$$G_{2u}(t, \vartheta) u_2 = \sum_{j=1}^{\infty} \omega_j(\cdot) \int_t^{\vartheta} \langle F(\omega_j), b_2 \rangle_{L^2(\Gamma)} e^{-\lambda_j(\vartheta - \tau)} u_2(\tau) d\tau$$

$$G_{2v}(t, \vartheta) v_2 = \sum_{j=1}^{\infty} \omega_j(\cdot) \int_t^{\vartheta} \langle F(\omega_j), c_2 \rangle_{L^2(\Gamma)} e^{-\lambda_j(\vartheta - \tau)} v_2(\tau) d\tau$$

$$\gamma(t, y, \vartheta) = \sup_{\|h\|_{L^2(\Omega)} = 1} \varphi(t, y, \vartheta, h)$$

$$\varphi(t, y, \vartheta, h) = \rho_{1v}(t, \vartheta, h) + \rho_{2v}(t, \vartheta, h) - \rho_{1u}(t, \vartheta, h) - \rho_{2u}(t, \vartheta, h) + \rho_M(h) - \langle h, G_1(t, \vartheta) f\rangle_{L^2(\Omega)} - \langle h, G_0(t, \vartheta) y\rangle_{L^2(\Omega)}$$

$$\rho_{iv}(t, \vartheta, h) = \max_{\{v_i\}} \langle h, G_{iv}(t, \vartheta) v_i\rangle_{L^2(\Omega)}$$

$$\rho_{iu}(t, \vartheta, h) = \max_{\{u_i\}} \langle h, G_{iu}(t, \vartheta) u_i\rangle_{L^2(\Omega)}$$

$$\rho_M(h) = \min_{q \in M(\vartheta)} \langle h, q\rangle_{L^2(\Omega)}$$

Condition 4.1. If the inequality $\gamma(t, y, \vartheta) > 0$ is satisfied for certain $t \in [t_0, \vartheta)$ and $y \in L^2(\Omega)$, the upper bound in (4.1) is reached on a single element.

Condition 4.1 is an analog of the regularity condition [1] (see also [8]). The following theorem holds:

Theorem 4.1. Let Condition 4.1 be satisfied. The element $\{t, y\} \in W(M, L^2(\Omega))$ if and only if $\gamma(t, y, \vartheta) \leqslant 0$.

The proof of Theorem 4.1 follows the plan of the proof of the similar statement in [8].

Note 4.1. Condition 4.1 is satisfied knowingly if for each $t \in [t_0, \vartheta)$ the functional

$$\chi(t, \vartheta, h) = \rho_{1v}(t, \vartheta, h) + \rho_{2v}(t, \vartheta, h) - \rho_{1u}(t, \vartheta, h) - \rho_{2u}(t, \vartheta, h) + \rho_M(h)$$

is concave in h. In its own turn, the concavity of $\chi(t, \vartheta, h)$ for any t holds if, for example, the following uniformity condition is satisfied: concave sets $R_i(t) \subset L^2(\Omega)$, $i = 1, 2$, exist for each $t \in [t_0, \vartheta)$, such that for almost all $t \in [t_0, \vartheta)$

$$G_{iu}(t, \vartheta) P_i(t) = G_{iv}(t, \vartheta) Q_i(t) + R_i(t), \quad i = 1, 2$$

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Translated by N. H. C.